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Numerical Solutions of the Compressible Hodograph Equation

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SOLUTION of the hodograph equation has not been extensively explored for engineering purpose, even though this equation is linear, since the final physical configurations corresponding to these indirect solutions are often not of practical interest. However, a few configurations which assume shapes of straight-line profiles even in the two-dimensional geometry do have practical importance as a propulsive or metering device. Figure 1 shows one of such typical configurations. Early solutions of these problems for incompressible flow were obtained from free streamline theory with conformal mapping by von Mises.[1] The corresponding problem of compressible flow issuing from an orifice was solved by Busemann[2] by introducing the tangent gas approximation. In the interest of considering conical convergent nozzles as the primary propulsive device of an ejector system, the corresponding isoclines (line of constant flow angle) in the vicinity of the throat were approximated by Brown[3] and Brown and Chow[4] as obtained from the corresponding two-dimensional solution, which was again derived from conformal mapping after the tangent gas relationship was introduced. The sonic line for such a nozzle flow subsequently was established by calculations from the method of characteristics. These studies also clarified the choked or unchoked flows, depending upon whether the back pressure would influence the establishment of the sonic line. This method of analysis has been employed by Anderson[5,6] to evaluate the performance of aircraft ejector propulsive systems. Excellent agreement between the theoretical results and the experimental data clearly indicates that the approach is adequate for practical applications.

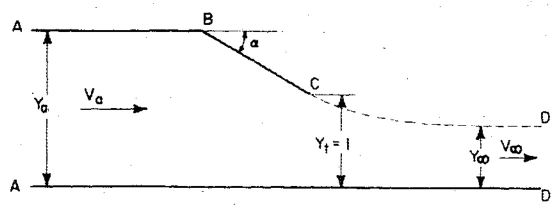

Fig. 1 Typical primary nozzle of an ejector system.

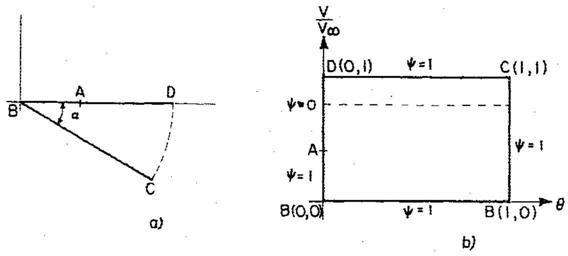

Fig. 2 The hodograph.

It is the intention of this Note to show that the compressible hodograph equation given by[7]

$$V^2\psi_{VV} + V(1+M^2)\psi_V + (1-M^2)/\alpha^2\psi_{\theta\theta} = 0 \tag{1}$$

can be solved by numerical calculations for this type of problem. Variables ψ, V, and θ in Eq. (1) are already normalized by the respective reference quantities so that their range of variation is from zero to one.

It may be easily seen from Fig. 2 that the boundary values of ψ for the reactangular domain of V and θ are completely specified. The correct values of ψ within the domain may be established by the well-known successive over-relaxation scheme. Calculations may be terminated when the variation of ψ is less than an arbitrarily small value (e.g., 1.0×10^{-6}). Once this is established, the partial derivative of ψ_V and ψ_θ may be evaluated for all points including that on the boundary and the solution is interpreted back to the physical plane through integrating the following system of differential

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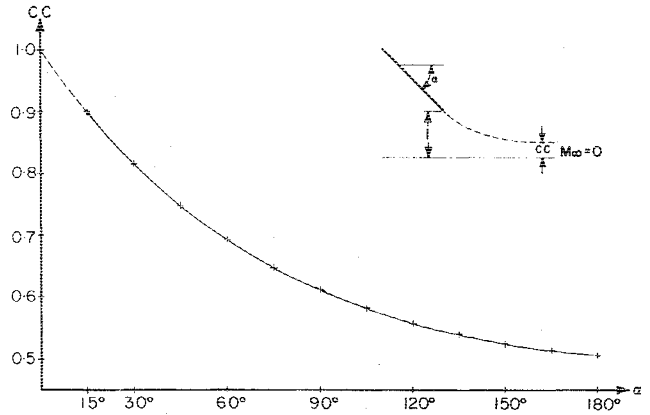

Fig. 3 Results of incompressible numerical calculation ($V_a = 0$).

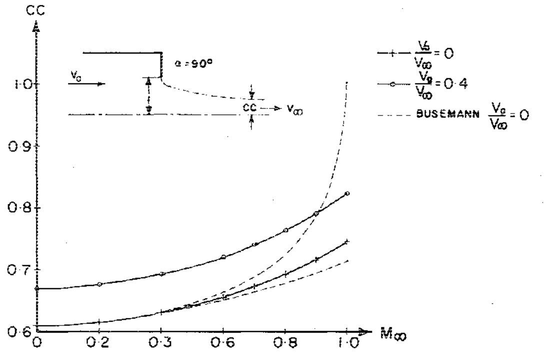

Fig. 4 Contracting coefficient for compressible flow ($\alpha = 90$ deg).

equations[7]:

$$d\left(\frac{x}{L}\right) = \left(\frac{\cos(-\alpha\theta)}{V}\phi_V - \frac{\rho_0}{\rho}\frac{\sin(-\alpha\theta)}{V}\psi_V\right)dV + \left(\frac{\cos(-\alpha\theta)}{V}\phi_\theta - \frac{\rho_0}{\rho}\frac{\sin(-\alpha\theta)}{V}\psi_\theta\right)d\theta \tag{2}$$

$$d\left(\frac{y}{L}\right) = \left(\frac{\sin(-\alpha\theta)}{V}\phi_V + \frac{\rho_0}{\rho}\frac{\cos(-\alpha\theta)}{V}\psi_V\right)dV + \left(\frac{\sin(-\alpha\theta)}{V}\phi_\theta + \frac{\rho_0}{\rho}\frac{\cos(-\alpha\theta)}{V}\psi_\theta\right)dV \tag{3}$$

where

$$\phi_V = (\rho_0/\rho)(I - M^2)/(V\alpha)\psi_\theta \tag{4}$$

$$\phi_\theta = -(\rho_0/\rho)V\alpha\psi_V \tag{5}$$

The reference length L (or the scale factor) is adjusted such that Y_t is unity (Fig. 1). This is accomplished by integrating Eq. (3) from point C first along the constant θ line to any intermediate velocity value and subsequently along the constant velocity line as shown by the dotted line in Fig. 2 until the horizontal axis in the physical plane is reached. The asymptotic height of the jet and the contracting coefficient can be obtained by calculating the freejet boundary starting from point C. It should be remarked that the final asymptotic state occurs only when x appraoches infinity. This may be observed from the fact that ψ_V approaches infinity at the asymptotic state D. However, $\sin\theta\,\psi_V$ approaches zero at this state so that a finite contracting coefficient results as it should.

Figure 3 shows the contracting coefficient for incompressible flow with negligible approaching flow velocity. The agreement with the exact solution, generally within a fraction of 1% indicates the merit of these calculations. Figure 4 presents the contracting coefficient for compressible flow discharging from an orifice ($\alpha = 90$ deg) for $V_a/V_\infty = 0$ and 0.4. Busemann's original results for $V_a/V_\infty = 0$ from tangent gas approximation are also shown in the same figure. Results for any angle α with different approaching flow velocities can easily be produced. With a relaxation factor of 1.25, one typical set of complete flow calculation takes 0.5 s on the CYBER 175 computing system.

For high-pressure ratios such that the freejet flow is supersonic, additional calculations based on the method of characteristics as suggested by Brown[3] may be performed to produce the downstream freejet flowfield and the isoclines obtained from sonic outflow conditions may be employed for these calculations. It is believed that compressible flow calculations for nonsymmetric configurations such as that studied by von Mises for incompressible flow can also be performed by the present scheme of calculations.

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Hypersonic Viscous Shock-Layer Flow over a Highly Cooled Sphere

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Nomenclature

a,b,c,d,e = coefficients of finite-difference equations
C_f = skin-friction coefficient, $2\tau_w^*/[\rho_\infty^*(U_\infty^*)^2]$

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